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ON J-CONVEXITY AND SOME ERGODIC SUPER-PROPERTIES OF BANACH SPACES

BY

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ABSTRACT. Given two Banach spaces $F|\ |$ and $X\|\ \|$, write F fr X iff for each finite-dimensional subspace F' of F and each number $\epsilon > 0$, there is an isomorphism V of F' into X such that $||x| - \|Vx\|| \leqslant \epsilon$ for each x in the unit ball of F'. Given a property P of Banach spaces, X is called super-P iff F fr X implies F is P. Ergodicity and stability were defined in our articles *On B-convex Banach spaces*, Math. Systems Theory 7 (1974), 294–299, and C. R. Acad. Sci. Paris Ser. A 275 (1972), 993, where it is shown that super-ergodicity and super-stability are equivalent to super-reflexivity introduced by R. C. James [Canad. J. Math. 24 (1972), 896–904]. Q-ergodicity is defined, and it is proved that super-Q-ergodicity is another property equivalent with super-reflexivity. A new proof is given of the theorem that J-spaces are reflexive [Schaffer-Sundaresan, Math. Ann. 184 (1970), 163–168]. It is shown that if a Banach space X is B-convex, then each bounded sequence in X contains a subsequence (y_n) such that the Cesàro averages of $(-1)^i y_i$ converge to zero.

Given two Banach spaces $F|\ |$ and $X\|\ \|$, F is said to be *finitely representable* in X, in symbols F fr X, iff for each finite-dimensional subspace F' of F and each number $\epsilon > 0$, there is an isomorphism V of F' into X such that $||x| - \|Vx\|| \leqslant \epsilon$ for each x in the unit ball of F'. Given a property P of Banach spaces, we say that X is *super*-P iff F fr X implies that F has the property P. Super-reflexive spaces were introduced by James [12], [13]; the result announced in [4] but implicit in the earlier paper [3] is that the following super-properties are equivalent: Super-ergodicity, super-reflexivity, super-Banach-Saks, super-stability. Here we define Q-ergodicity, a notion in appearance weaker than ergodicity, and prove that super-Q-ergodicity is another property equivalent with super-reflexivity. At the same time we give a new proof of James's theorem [10] that $(2, \epsilon)$-convex spaces are reflexive, and more generally of the recent results of Schaffer-Sundaresan [19], that J-spaces are reflexive. We also show that

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B-convex spaces are *alternate signs Banach-Saks*: Each bounded sequence contains a subsequence (y_n) such that the Cesàro averages of $(-1)^i y_i$ converge to zero.

1. **Preliminaries.** Let X be an arbitrary Banach space with norm $\| \ \|$. An *isometry* (*contraction*) is a linear map $T\colon X \longrightarrow X$ such that $\|Tx\| = \|x\|$ ($\|Tx\| \leqslant \|x\|$) for each $x \subset X$. The Cesàro averages $(1/n)(T^0 + \cdots + T^{n-1})$ are denoted by A_n, or $A_n(T)$. The following simple result seems new.

PROPOSITION 1.1. *If T is a contraction on a Banach space X then for each $x \in X$ the limit of $\|A_n x\|$ exists.*

PROOF. Let $x \in X$ and set $\alpha = \liminf \|A_n x\|$. It suffices to show that, for each $\delta > 0$,

$$\limsup \|A_n x\| \leqslant \alpha + \delta. \tag{1.1}$$

Given a $\delta > 0$, choose a *fixed* integer N such that $\|A_N x\| \leqslant \alpha + \delta$. If m and n are positive integers, $mN \leqslant n < (m+1)N$, then as $m \longrightarrow \infty$

$$\|A_{mN} - A_n\| \longrightarrow 0. \tag{1.2}$$

Therefore it suffices to prove that $\limsup_m \|A_{mN} x\| \leqslant \alpha + \delta$. $\|T\| \leqslant 1$ implies that $\|T^{jN} A_N x\| \leqslant \alpha + \delta$ for each j. Hence for each m

$$\|A_{mN}(T)x\| = \|A_m(T^N) A_N(T)x\| \leqslant \alpha + \delta. \tag{1.3}$$

This proves that $\lim \|A_n x\|$ exists. It is easy to see that this limit, considered as a function of x, is a seminorm. □

Note that applying the proposition to the space of bounded operators on X one obtains: for each contraction T, $\lim \|A_n(T)\|$ exists.

We will now define Q-ergodicity. Let S be the space of all sequences $a = (a_i)_{i=1,2,\cdots}$ such that $a_i = 0$ but all but finitely many i's. Assuming T fixed, set, for each $x \in X$ and $a \in S$,

$$\begin{aligned} &Q(x; a; n_1, n_2, \cdots) \\ &\quad = \|a_1 A_{n_1} x + a_2 A_{n_2} T^{n_1} x + a_3 A_{n_3} T^{n_1+n_2} x + \cdots \|, \end{aligned} \tag{1.4}$$

$$L(x, a) = \limsup_{n \to \infty} Q(x; a; n_1, n_2, \cdots), \qquad n = \inf(n_i). \tag{1.5}$$

The lim sup above becomes *limit* if a is one-dimensional (by Proposition 1.1), or if the norm is "invariant under spreading of the sequence $T^n x$" (see Proposition 2.2 below).

Let r be an integer $\geqslant 2$ and ϵ a number, $0 \leqslant \epsilon \leqslant 1/r$. The space X is called (r, ϵ)-*ergodic* iff for each isometry T, each $x \in X$, any r elements

$a^1, \cdots, a^r$ of S such that $L(x, a^i) \leqslant 1$, one has

$$\min_{1 \leqslant k \leqslant r} L(x, a^1 + a^2 + \cdots + a^k - a^{k+1} - a^{k+2} - \cdots - a^r) \leqslant r(1-\epsilon). \tag{1.6}$$

X is called *Q-ergodic*, or *qualitatively ergodic*, iff it is ergodic for some r and ϵ. We recall that X is called *ergodic* (for isometries) iff $\lim A_n x$ exists for each isometry T and each $x \in X$. We now will show that if X is ergodic, then it is (r, ϵ)-ergodic for each r and ϵ. It is known and easy to see that the ergodic theorem for T implies that, for each x, $\lim_n A_n T^j x$ exists uniformly in j. (Apply, e.g., the decomposition theorem [6, p. 662]; uniform in j converges to the limit is obvious for a T-invariant x, and also for an x of the form $x = y - Ty$.) Let $\bar{x} = \lim A_n x$, $a^j = (a_i^j)$, $\alpha_j = \Sigma_i a_i^j \|\bar{x}\|$. If T is ergodic then the jth summand in (1.4) converges to $a_j \bar{x}$, hence (1.6) follows from the inequality

$$\begin{aligned} \min_{1 \leqslant k \leqslant r} |\alpha_1 + \alpha_2 + \cdots + \alpha_k - \alpha_{k+1} - \cdots - \alpha_r| &\leqslant (r-1) \sup_j |\alpha_j| \\ &\leqslant r(1-\epsilon), \end{aligned} \tag{1.7}$$

easy to verify by induction on r.

A Banach space X is called J-(r, ϵ)-convex, where $r \geqslant 2$, $0 \leqslant \epsilon < 1$, iff for each r-tuple $(x_1, \cdots, x_r)$ of elements of the unit ball U_X of X one has

$$\min_{1 \leqslant k \leqslant r} \|x_1 + \cdots + x_k - x_{k+1} - \cdots - x_r\| \leqslant r(1-\epsilon). \tag{1.8}$$

X is called *J-convex* iff it is J-(r, ϵ)-convex for some r and ϵ. It follows from a recent unpublished result of R. C. James [13] that J-convexity is a properly stronger notion than B-convexity introduced in [2]; cf. §3 below. It is easy to see that J-(r, ϵ)-convexity, hence J-convexity, are super-properties; i.e., if X enjoys them, so does every space finitely representable in X. It has been proven by Schaffer-Sunderasan [19], and will be again shown below, that J-convex spaces are reflexive; hence, as already noted in [14], super-reflexive. Since the ergodic theorem holds for reflexive spaces, it follows that J-convex spaces are ergodic. It would be perhaps of interest to give a direct proof of this result; here we only point out that J-$(2, \epsilon)$-convexity easily implies the relation:

$$\limsup_{n,p \to \infty} \|A_n(T)x - A_p(T)x\| \leqslant 2(1-\epsilon) \lim \|A_n(T)x\| \tag{1.9}$$

for each contraction T on X and each $x \in X$: Note that for any fixed positive integers i, N, m one has the identities

$$A_{2iN} = A_i(T^N)\left[\tfrac{1}{2}(A_N + T^{iN} A_N)\right], \tag{1.10}$$

$$A_N - A_{mN} = \frac{1}{m} \sum_{i=1}^{m-1} (A_N - T^{iN} A_N). \tag{1.11}$$

Let $x \in X$; $\lim \|A_n x\| = \alpha$ exists by Proposition 1.1. Select a fixed number δ, $0 < \delta < \epsilon\alpha/(2-\epsilon)$. Choose a fixed N so large that

$$| \|A_{N+k} x\| - \alpha | < \delta, \qquad k = 0, 1, \cdots . \tag{1.12}$$

Since $\|T\| \leqslant 1$, either for some integer i

$$\|A_N x + T^{iN} A_N x\| \leqslant 2(1-\epsilon)(\alpha + \delta), \tag{1.13}$$

or for all i

$$\|A_N x - T^{iN} A_N x\| \leqslant 2(1-\epsilon)(\alpha + \delta). \tag{1.14}$$

In the first case (1.10) implies $\|A_{2iN} x\| \leqslant (1-\epsilon)(\alpha+\delta)$, which contradicts (1.12). Therefore (1.14) must hold for all i, and (1.11) implies $\|A_N x - A_{mN} x\| \leqslant 2(1-\epsilon)(\alpha+\delta)$ for all m. Since δ may be chosen arbitrarily small, (1.2) now implies (1.9).

2. **Ergodic super-properties.** A Banach space X with norm $\| \ \|$ is given. A bounded sequence (x_n) in X is called *stable* iff there is an element $\bar{x}$ such that

$$\lim_n \left\| \frac{1}{n} \sum_{i=1}^{n} x_{k_i} - \bar{x} \right\| = 0 \tag{2.1}$$

uniformly in the set K of all strictly increasing sequences (k_n) of natural numbers. Actually, the *uniformity* is an easy consequence of convergence for *all* $(k_n) \in K$. A Banach space is called *stable* iff every bounded sequence contains a stable subsequence; *Banach-Saks* iff every bounded sequence contains a subsequence which converges Cesàro. Professor Paul Erdös has recently informed us that he had shown jointly with Professor M. Magidor that every space which is Banach-Saks is also stable, the proof being based on the combinatorial fact that every analytic set is Ramsey [20].

We now return to the setting of our papers [3], [4], in which we have attempted to connect ergodic properties of X with stability, or the Banach-Saks property. We have at first asked the following question: Does an arbitrary bounded sequence (x_n) in X admit a subsequence (e_n) such that the shift T on (e_n) is defined and power-bounded? (By a shift on (e_n) we understand an operator T satisfying $Te_n = e_{n+1}$ for all n, and acting on the space spanned by the e_n's.) If the answer to this question had been *positive*, it would follow at once that the ergodic theorem (power-bounded version) for X and its subspaces implies the Banach-Saks property—therefore the answer is *negative*, since there are reflexive spaces which are not Banach-Saks (Baernstein [1]). This showed the need to change the norm. Denoting the space spanned by (e_n) and a new norm $| \ |$ by F, we could obtain [3] that the shift on (e_n) be an isometry, and yet $| \ |$ be so close to $\| \ \|$ that the ergodic theorem for T on F implies that (e_n) contains a stable subsequence in X, and F fr X. The implication announced in [4], super-ergodic $\Rightarrow$ super-stable, follows. We recapitulate the construction of (e_n) and F. S is the space of all sequences $a = (a_i)_{i=1,2,\cdots}$ with $a_i = 0$ for all but finitely many i. We have

PROPOSITION 2.1 (PROPOSITION 1 OF [3]). *Each bounded sequence (x_n) in X contains a subsequence (e_n) with the following property: For each $a \in S$ there exists a number $L(a)$ such that $\| \Sigma a_i e_{n_i} \| \longrightarrow L(a)$ as the sequences $(n_1), (n_2), \cdots$ converge to ∞ so that $n_1 < n_2 < \cdots$.*

Now fix (x_n) and let (e_n) be a subsequence of (x_n) satisfying the conditions of the above proposition. Let $\varphi(S)$ be the space of linear combinations $\Sigma a_i e_i$, $a \in S$. As shown in [3], we may assume without loss of generality that the e_n's are algebraically independent in X, and that $|\Sigma a_i e_i|$ defined as equal to $L(a)$ is a *norm* on $\varphi(S)$. We denote by F the completion of $\varphi(S)$ in this norm. We now show that F fr X: If F' is an n-dimensional subspace of F, F' is topologically isomorphic to $l_1^{(n)}$, hence we commit a negligible error assuming that F' is generated by $e_1, \cdots, e_m$ for m large. Let the same vectors in X generate a subspace H. Set $S_n = T^n: H \longrightarrow X$. Then $\|S_n x\| \longrightarrow |x|$ on H implies $M = \sup_n \|S_n\| < \infty$ (uniform boundedness principle), hence $\|T^n x\| \longrightarrow |x|$ uniformly on compacts of H; therefore uniformly on $U_{F'}$. Indeed, if $Y = \{y_i\}$ is a *finite* δ-net in a compact $C \subset H$, then $| \|T^n x\| - |x| | < \delta$ on Y implies that $| \|T^n x\| - |x| | \leqslant \delta + 2\delta M$ on C. To see this, note that if $\|x - y_i\| < \delta$ then

$$| \|T^n x\| - |x| | \leqslant | \|T^n x\| - \|T^n y_i\| | + | \|T^n y_i\| - |y_i| | + | |y_i| \ \ |x| - $$
$$\leqslant \delta M + \delta M.$$

The relation F fr X was already implicitly used in Lemma 6 [3] and in [4]. Parting from F we now propose to introduce a new norm ! ! on (e_n), with properties even more pleasing than $| \ |$; the space G generated by (e_n), ! ! will still be finitely representable in X. The main virtue of $| \ |$ (not included in isometric character of the shift T) may be described as *invariance under spreading*, or (IS) property: The norm of any finite combination of the e_n's remains the same when the vectors are shifted, even though their mutual distances (but not positions) may change. This property, formally stated in [3, Lemma 1], is an immediate consequence of Proposition 2.1. The norm ! ! will inherit from | | the (IS) property, but will also be *equal signs additive*, in short of type (ESA): In computing the norm of any finite linear combination of the e_i's, consecutive terms of equal sign may be combined. Formally, for any vector $x = a_1e_1 + \cdots + a_qe_q$, any integers k, p such that $1 \leqslant k < p \leqslant q$ and $a_i \geqslant 0$ for $k \leqslant i \leqslant p$, one has

$$!x! = \left|\sum_{i=1}^{k-1} a_ie_i + (a_k + \cdots + a_p)e_k + \sum_{i=p+1}^{q} a_ie_i\right|. \tag{2.2}$$

It is easy to see that it suffices to verify (2.2) for all k and p such that $p - k = 1$.

We now let $A_n(T)$ act on the e_i's spread so that different averages have disjoint support. More precisely, given a fixed $a = (a_i) \in S$ with $a_i = 0$ for $i > q$, we define

$$\begin{aligned} &P(n_1, \cdots, n_q; s_1, \cdots, s_q) = a_1A_{n_1}e_{s_1} + \cdots + a_qA_{n_q}e_{s_q}, \\ &\qquad s_1 > 0,\ s_2 \geqslant s_1 + n_1, \cdots, s_q \geqslant s_{q-1} + n_{q-1}. \end{aligned} \tag{2.3}$$

Invariance of | | under spreading implies that the F-norm of the first expression in (2.3) does not depend upon the choice of the s_i's; therefore this norm will be denoted by $Q(n_1, \cdots, n_q)$, or $Q(e_1; a; n_1, \cdots, n_q)$.

PROPOSITION 2.2. *For each $x = a_1e_1 + \cdots + a_qe_q$ in $\varphi(S)$, the limit of $Q(e_1; a; n_1, \cdots, n_q)$ as $\inf(n_i)$ converges to infinity exists. This limit, denoted $!x!$, is a seminorm on $\varphi(S)$.*

PROOF. The invariance of | | under spreading implies that for any fixed positive integers $N_1, \cdots, N_q; m_1, \cdots, m_q$

$$Q(m_1N_1, \cdots, m_qN_q) \leqslant Q(N_1, \cdots, N_q). \tag{2.4}$$

The particular case of (2.4) where $m_i = 1$ for $i = 2, \cdots, q$ is obtained by taking the Cesàro average of

$$P(a_1T^{kN_1}A_{N_1}e_{s_1} + a_2A_{N_2}e_{s_2} + \cdots + a_qA_{N_q}e_{s_q})$$

$$\text{for } k = 0, 1, \cdots, m_1 - 1,$$

since $s_2 > s_1 + m_1N_1$, $s_i \geqslant s_{i-1} + N_i$ for $i = 3, \cdots, q$ implies that each term has the norm $= Q(N_1, \cdots, N_q)$. An obvious induction argument, again using invariance under spreading of | |, establishes (2.4). We denote by α (β) the limit inferior (limit superior) of $Q(n_1, \cdots, n_q)$ as n_i converge independently to infinity. To prove the proposition, it suffices to show that, for each $\delta > 0$, $\beta \leqslant \alpha + \delta$. Choose $N_1, \cdots, N_r$ *fixed* such that $Q(N_1, \cdots, N_q) \leqslant \alpha + \delta$; (2.4) implies $Q(m_1N_1, \cdots, m_rN_r) \leqslant \alpha + \delta$ for all m_i. A computation analogous to (1.2) shows that if $m_i N_i \leqslant n_i < (m_i + 1)N_i$ for all i, then

$$\limsup_{m_i \to \infty} |Q(m_1 N_1, \cdots, m_q N_q) - Q(n_1, \cdots, n_q)| = 0. \tag{2.5}$$

$\beta \leqslant \alpha + \delta$ follows. Finally, it is easy to see that $!\ !$ is a seminorm on $\varphi(S)$.

LEMMA 2.1. *The seminorm* $!\ !$ *is of type* (ESA) *on the* e_n*'s.*

PROOF. We verify (2.2) assuming, as we may, that $p = k + 1$. Since $!\ !$ is a continuous function of coefficients a_i, we further may suppose that a_k/a_{k+1} is a rational number, and write $a_k = \alpha r$, $a_{k+1} = \alpha s$, where r, s are positive integers. Then for all integers $m > 0$, $t \geqslant 0$, one has

$$a_k A_{mr} e_t + a_{k+1} A_{ms} e_{mr+t} = (a_k + a_{k+1}) A_{m(r+s)} e_t. \tag{2.6}$$

The relation (2.5) is now applied, with $m_k = m_{k+1} = m$, $N_k = r$, $N_{k+1} = s$ to compute $!x!$, and with $N_k = r + s$ and $m_k = m$ to compute the right-hand side of (2.2) which is thus established. $\square$

LEMMA 2.2. *If* $!e_1 - e_2! = 0$ *then* (e_n) *admits a subsequence stable in* X.

PROOF. $!e_1 - e_2! = 0$ implies that

$$|A_n e_1 - A_r e_{1+n}| \longrightarrow 0 \quad \text{and} \quad |A_p e_1 - A_r e_{1+p}| \longrightarrow 0 \quad \text{as } n, p, r \longrightarrow \infty.$$

Choosing r so that $r/(n + p) \longrightarrow \infty$, we have that $|A_r e_{1+n} - A_r e_{1+p}| \longrightarrow 0$; therefore by the triangular inequality the sequence $A_n e_1$ is Cauchy in F. Proposition 3 [3] is now applicable. $\square$

Since we wish to prove that the space X is stable, we only need to consider the case when $!e_1 - e_2! > 0$; then $!\ !$ may be easily seen to be a *norm* on $\varphi(S)$: If $!\Sigma_{i=1}^q a_i e_i! = 0$, then $!a_1 e_1 + a_3 e_3 + \cdots + a_{q+1} e_{q+1}! = 0$ and also $!a_1 e_2 + a_3 e_3 + \cdots + a_{q+1} e_{q+1}! = 0$; hence $!a_1(e_1 - e_2)! = 0$ which implies $a_1 = 0$. Similarly one shows that $a_2 = 0, \cdots, a_q = 0$. Denote by G the completion of $\varphi(S)$ in this norm.

We show that G is finitely representable in F, hence in X. Let G' be a finite-dimensional subspace of G; we may assume that G' is generated by e_1, $e_2, \cdots, e_q$. Let $V = V_{n_1, \cdots, n_q}$ map each vector $a_1 e_1 + \cdots + a_q e_q$ onto

$$a_1 A_{n_1} e_1 + a_2 A_{n_2} T^{n_1} e_1 + \cdots + a_q A_{n_q} T^{n_1 + \cdots + n_{q-1}} e_1.$$

Then for all $x \in U_{G'}$, by Proposition 2.2 $|!x! - |Vx||$ is small if $n_1, \cdots$, n_q are large. G fr F easily follows (see the proof of F fr X above).

Define a seminorm M on S by

$$M(a) = !a_1(e_1 - e_2) + a_2(e_3 - e_4) + a_3(e_5 - e_6) + \cdots !. \tag{2.7}$$

REMARK. The proofs of the following Lemma 2.3 and Proposition 2.3 use only the (IS) property of the norm. Thus they remain valid with $|\ |$ replacing $!\ !$.

LEMMA 2.3. *$M(a) \geqslant M(b)$ if for each i, $a_ib_i \geqslant 0$ and $|a_i| \geqslant |b_i|$. Hence M is orthogonal, i.e., $M(a) \geqslant M(a^+)$, $M(a) \geqslant M(a^-)$, where a^+ is a sequence (a_i^+), a^- the sequence (a_i^-).*

PROOF. The invariance under spreading of $|\ |$ implies that for each j, each n,

$$\begin{aligned} M(a) &= !y + a_j(e_{2j-1} - e_{2j}) + z! = !y + a_j(e_{2j} - e_{2j+1}) + z! \\ &= \cdots = !y + a_j(e_{2j-1+n} - e_{2j+n}) + z! \end{aligned} \tag{2.8}$$

where

$$y = \sum_{i=1}^{j-1} a_i(e_{2i-1} - e_{2i}), \qquad z = \sum_{i=j+1}^{\infty} a_i(e_{2i+n-1} - e_{2i+n}). \tag{2.9}$$

(Since $a \in S$, z has only finitely many summands.) Summing the $n + 1$ expressions inside $!\ !$ in (2.8) and dividing by $n + 1$, one obtains

$$M(a) \geqslant !y + z! - |a_j|!(e_{2j-1} - e_{2j+n})/(n + 1)!.$$

Let $n \longrightarrow \infty$; it follows that $M(a) \geqslant M(a')$, where $a_j' = 0$, $a_i' = a_i$ for $i \neq j$. The lemma is proved, because M is a convex function of coordinates.

PROPOSITION 2.3. *If G does not contain an isomorphic copy of c_0, then*

$$\lim_n !e_1 - e_2 + e_3 - e_4 + \cdots + e_{2n-1} - e_{2n}! = \infty. \tag{2.10}$$

PROOF. Set $u_n = e_{2n-1} - e_{2n}$; let G' be the subspace of G generated by the u_i's. Write $M(\Sigma a_iu_i) = M(a)$ for $a \in S$; extended to G', M is a norm coinciding with $!\ !$. Let $|a| = a^+ + a^-$, $N(a) = M(|a|)$, $N(y) = N(a)$ if $y = \Sigma a_iu_i$, $a \in S$. $N(a) \leqslant M(a^+) + M(a^-) \leqslant 2M(a)$ by Lemma 2.3. Therefore

$$\tfrac{1}{2}N(a) \leqslant M(a) \leqslant N(a). \tag{2.11}$$

Extended to G', N is a norm equivalent with M. This observation will be useful in §3 below. Now if (2.10) fails, Lemma 2.3 gives a β such that, for all n,

$$!e_1 - e_2 + e_3 - e_4 + \cdots + e_{2n-1} - e_{2n}! \leqslant \beta, \tag{2.12}$$

and also shows that $M(a^+) \leqslant \beta \cdot \sup(|a_i|)$, $M(a^-) \leqslant \beta \sup(|a_i|)$, $M(a) \leqslant 2\beta \sup(|a_i|)$. Also, $M(a) \geqslant !a_i(e_{2i-1} - e_{2i})! = |a_i|(e_1 - e_2)$, so that $M(a) \geqslant$

$(\sup|a_i|)\|e_1 - e_2\|$. Thus G' is a subspace of G that is isomorphic to c_0.

PROPOSITION 2.4. *If* (2.10) *holds, then* G *is not J-convex.*

PROOF. We show that G is not J-(r, ϵ)-convex by first giving a detailed and "graphic" proof of the case $r = 2$, then a brief proof of the general case. Set

$$v_n = e_1 \quad - e_3 \quad + \cdots + e_{4n-3} \quad - e_{4n-1} \quad ,$$

$$w_n = \quad + e_2 \quad - e_4 + \cdots + \quad e_{4n-2} \quad - e_{4n}.$$

We have $\|v_n\| = \|w_n\| = \|v_n + w_n\|/2$, the last equality by (ESA). To prove that G is not J-$(2, \epsilon)$-convex, it will suffice to prove that $\|v_n - w_n\|/2\|v_n\|$ converges to 1. This follows from (2.10) because

$$\begin{aligned}\|v_n - w_n\| = \|e_1 &- (e_2 + e_3) + (e_4 + e_5) - \cdots \\ &- (e_{4n-2} + e_{4n-1}) + (e_{4n} + e_{4n+1}) - e_{4n+1}\| \\ &\geq - \|e_1\| + 2\|v_n\| - \|e_{4n+1}\|.\end{aligned}$$

We now show, by essentially the same argument, that G is not J-(r, ϵ)-convex, where $r \geq 2$ is arbitrary. Set for $j = 1, 2, \cdots, r; n = 1, 2, \cdots$,

$$v_n^j = e_j - e_{j+r} + e_{j+2r} - \cdots + e_{j+(2n-2)r} - e_{j+(2n-1)r}. \tag{2.13}$$

Then $\|v_n^j\| = \|v_n^1\|$ for each j. In the expression $d_n^k = v_n^1 + v_n^2 + \cdots + v_n^k - v_n^{k+1} - \cdots - v_n^r$ the terms are arranged as follows: First write $S_1 = e_1 + e_2 + \cdots + e_k$. Then $(2n - 1)r$ terms grouped so that r consecutive e_i's with $-$ sign alternate with r consecutive e_i's with $+$ sign:

$$\begin{aligned}S_2 = - (e_{k+1} + e_{k+2} + \cdots + e_{k+r}) + (e_{k+r+1} + \cdots + e_{k+2r}) - \cdots \\ + (e_{k+2(n-2)r+1} + \cdots + e_{k+2(n-1)r}).\end{aligned}$$

S_3 is composed of the remaining $r - k$ terms of d_n^k. Then $\lim_n \|S_i\|/r\|v_n^1\| = 0$ for $i = 1, 3$; $= 1$ for $i = 2$. Hence $\lim_n \|d_n^k\|/r\|v_n^1\| = 1$ for each $k = 1, 2, \cdots, r$. The proposition is proved.

Now assume that X is J-convex; then so is G and G cannot contain an isomorphic copy of c_0 (cf. [10] or [8], where this is proved for B-convex spaces). Propositions 2.3 and 2.4 and Lemma 2.2 now imply the following theorem:

THEOREM 2.1. *A J-convex Banach space is stable (hence super-stable).*

THEOREM 2.2. *A super-Q-ergodic Banach space is super-stable.*

PROOF. If X is super-Q-ergodic then G is Q-ergodic, and the proof of Proposition 2.4 yields a contradiction. Lemma 2.2 now implies that (x_n) has a subsequence stable in X; since (x_n) is arbitrary, it follows that X is stable. Thus super-Q-ergodicity implies stability; it implies super-stability because the relation "fr" is transitive. □

Since a Banach-Saks space, and *a fortiori* a stable space, is easily seen to be reflexive (cf. [17]), the argument above provides a new proof that J-convex spaces are reflexive. We finally observe that in the course of the proof of Theorem 2.1 we establish the following: Any sequence (x_n) admits a subsequence (e_n) such that the sequence $(e_{2n-1} - e_{2n})$ is an unconditional basis for the IS norm $|\ |$, finitely representable in $\|\ \|$. (Because an orthogonal norm is unconditional, and, as observed above, the proofs of Proposition 2.3 and Lemma 2.3 are valid for the norm $|\ |$ as well as $!\ !$.)

3. **Alternate signs Banach-Saks property.** A Banach space X is called *(r, ϵ)-convex* iff for any r elements $x_1, \cdots, x_r$ in U_X there is a sequence of signs $\sigma_1, \cdots, \sigma_r$ such that $(1/r)(\sigma_1 x_1 + \cdots + \sigma_r x_r) \leq 1 - \epsilon$. A Banach space is called *B-convex* iff it is (r, ϵ)-convex for some integer r and some $\epsilon > 0$.

THEOREM 3.1. *Every bounded sequence (x_n) in a B-convex Banach space admits a subsequence (y_n) such that*

$$\frac{1}{n}\sum_{i=1}^{n}(-1)^{i+1}y_i \longrightarrow 0. \tag{3.1}$$

PROOF. We may assume that (x_n) is not stable, since otherwise (y_n) satisfying (3.1) may be obtained as a union of two stable subsequences of (x_n). Let F' be the subspace of F generated by $u_1 = e_1 - e_2$, $u_2 = e_3 - e_4, \cdots$. If X is B-convex, then so is F' and therefore, as it is easy to see, there exists a sequence of signs (σ_n) such that

$$\liminf_n \left|\frac{1}{n}\sum_{i=1}^{n}\sigma_i u_i\right| = 0. \tag{3.2}$$

The proof of (3.2) is only sketched since the argument is known. We may assume $|u_i| \leq 1$ for all i. Let X be (r, ϵ)-convex. First choose signs $\sigma_1^1 = +, \sigma_2^1, \sigma_3^1, \cdots$ so that if $y_k = r^{-1}\Sigma_{i=1+kr}^{r+kr}\sigma_i^1 u_i$, then $|y_k| \leq 1 - \epsilon$ for $k = 0, 1, \cdots$. Second choose signs $\sigma_1^2 = +, \sigma_2^2, \sigma_3^2, \cdots$ so that if $z_k = r^{-1}\Sigma_{i=1+kr}^{r+kr}\sigma_i^2 y_i$ then $|z_k| \leq (1-\epsilon)^2$ for $k = 0, 1, \cdots$. Next take Cesàro averages of successive r-tuples of $\sigma_i^3 z_i$, where σ_i^3 are appropriate signs, etc. This procedure yields a sequence of signs σ_i satisfying (3.2).

As already observed, the proofs of Lemma 2.3 and Proposition 2.3, in particular (2.11), use only the (IS) property of the norm, hence remain valid with $|\ |$ replacing $!\ !$. Therefore (3.2) remains valid when all the σ_i's are replaced by the sign $+$. Proposition 1.1 with $-T$ replacing T now implies that

$n^{-1}(u_1 + \cdots + u_n)$ converges to zero in F. The proof of Proposition 3 [3] remains valid if (u_n) replaces (e_n); hence the sequence (u_n) contains a subsequence stable in X. This proves (3.1). $\square$

Applying the theorem that every analytic (or only Borel) set is Ramsey (cf. the remarks in the beginning of §2), one may strengthen Theorem 3.1 to read: Every bounded sequence (x_n) in a B-convex space contains a subsequence (z_n) such that (3.1) holds for each subsequence (y_n) of (z_n).

The alternate signs Banach-Saks property does *not* characterize B-convex spaces since c_0 has it, as has been shown to us by Professor A. Pełczyński.

PROPOSITION 3.1. *Let* (x_n) *be a sequence of vectors in* c_0, $x_n = (x_n^{(i)})_{i=1}^\infty$, *with* $\|x_n\| = \sup_i |x_n^{(i)}| \leqslant 1$ *for all* n. *Then for each* $\epsilon > 0$ *there exists a subsequence* (y_n) *of* (x_n) *such that for all integers* m

$$\left\| \sum_{j=1}^m (-1)^{j+1} y_j \right\| = \sup_i \left| \sum_{j=1}^m (-1)^{j+1} y_j^{(i)} \right| \leqslant 2 + \epsilon. \tag{3.3}$$

Hence (3.1) *holds.*

PROOF. Choose $\epsilon > 0$. Since we can pass to subsequences and apply the diagonal procedure, we may and do assume that $\lim_{n\to\infty} x_n^{(i)} = a_i$ exists for each i and also that $|x_n^{(i)} - a_i| < 2^{-n}\epsilon$ if $|x_k^{(i)}| > 2^{-k}\epsilon$ for some $k < n$. Then, for a subsequence (y_n),

$$\left\| \sum_{j=1}^m (-1)^{j+1} y_j \right\| = \sup_i \left| \sum_{j=1}^m (-1)^{j+1} y_j^{(i)} \right| < 2 + \epsilon,$$

since for each i we can replace by a_i each $x_n^{(i)}$ for which there exists $k < n$ such that $|x_k^{(i)}| > 2^{-k}\epsilon$, and obtain

$$\left| \sum_{j=1}^m (-1)^{j+1} y_j^{(i)} \right| < \epsilon \left(\sum_1^\infty 2^{-n} \right) + |x_k^{(i)}| + |a_i| \leqslant 2 + \epsilon. \ \square$$

Note that reflexive spaces need not be alternate signs Banach-Saks: The example in [1] is not alternate signs Banach-Saks.

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